Linear Algebra I

31/01/2017, Tuesday, 14:00 – 17:00

You are **NOT** allowed to use any type of calculators.

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1 Linear systems of equations (2 + 3 + 3 + 2 + 2 + 3 = 15 pts)

Consider the linear systems of equations

$$\begin{bmatrix} 2 & -3 & -1 & 2 & 3 \\ 4 & -4 & -1 & 4 & 11 \\ 2 & -5 & -2 & 2 & -1 \\ 0 & 2 & 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 9 \\ -5 \end{bmatrix}.$$

(a) Write down the corresponding augmented matrix.

(b) Put it into the row echelon form.

(c) Put it into the reduced row echelon form.

(d) What is the rank of the augmented matrix?

(e) Determine the lead and free variables.

(f) Find the solution set.

Required Knowledge: **Gauss-elimination, row operations, row echelon form, set of solutions, and rank.**

Solution:

1a: The augmented matrix is given by

$$\begin{bmatrix} 2 & -3 & -1 & 2 & 3 & \vdots & 4 \\ 4 & -4 & -1 & 4 & 11 & \vdots & 4 \\ 2 & -5 & -2 & 2 & -1 & \vdots & 9 \\ 0 & 2 & 1 & 0 & 4 & \vdots & -5 \end{bmatrix}.$$

1b: By applying row operations, we obtain:

$$\begin{bmatrix} 2 & -3 & -1 & 2 & 3 & \vdots & 4 \\ 4 & -4 & -1 & 4 & 11 & \vdots & 4 \\ 2 & -5 & -2 & 2 & -1 & \vdots & 9 \\ 0 & 2 & 1 & 0 & 4 & \vdots & -5 \end{bmatrix} \xrightarrow[\textbf{3rd} = \textbf{3rd} - \textbf{1st}]{\textbf{2nd} = \textbf{2nd} - 2 \times \textbf{1st}} \begin{bmatrix} 2 & -3 & -1 & 2 & 3 & \vdots & 4 \\ 0 & 2 & 1 & 0 & 5 & \vdots & -4 \\ 0 & -2 & -1 & 0 & -4 & \vdots & 5 \\ 0 & 2 & 1 & 0 & 4 & \vdots & -5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 & -1 & 2 & 3 & \vdots & 4 \\ 0 & 2 & 1 & 0 & 5 & \vdots & -4 \\ 0 & -2 & -1 & 0 & -4 & \vdots & 5 \\ 0 & 2 & 1 & 0 & 4 & \vdots & -5 \end{bmatrix} \xrightarrow[\mathbf{4th} = \mathbf{4th} - \mathbf{2nd}]{\mathbf{3rd} = \mathbf{3rd} + \mathbf{2nd}} \begin{bmatrix} 2 & -3 & -1 & 2 & 3 & \vdots & 4 \\ 0 & 2 & 1 & 0 & 5 & \vdots & -4 \\ 0 & 0 & 0 & 0 & 1 & \vdots & 1 \\ 0 & 0 & 0 & 0 & -1 & \vdots & -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 & -1 & 2 & 3 & \vdots & 4 \\ 0 & 2 & 1 & 0 & 5 & \vdots & -4 \\ 0 & 0 & 0 & 0 & 1 & \vdots & 1 \\ 0 & 0 & 0 & 0 & -1 & \vdots & -1 \end{bmatrix} \xrightarrow{\mathbf{4th} = \mathbf{4th} + \mathbf{3rd}} \begin{bmatrix} 2 & -3 & -1 & 2 & 3 & \vdots & 4 \\ 0 & 2 & 1 & 0 & 5 & \vdots & -4 \\ 0 & 0 & 0 & 0 & 1 & \vdots & 1 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Therefore, we obtain the row echelon form as follows:

$$\begin{bmatrix} 2 & -3 & -1 & 2 & 3 & \vdots & 4 \\ 0 & 2 & 1 & 0 & 5 & \vdots & -4 \\ 0 & 0 & 0 & 0 & 1 & \vdots & 1 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \xrightarrow[\mathbf{2nd} = \frac{1}{2} \times \mathbf{2nd}]{\mathbf{1st} = \frac{1}{2} \times \mathbf{1st}} \begin{bmatrix} 1 & -\frac{3}{2} & -\frac{1}{2} & 1 & \frac{3}{2} & \vdots & 2 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{5}{2} & \vdots & -2 \\ 0 & 0 & 0 & 0 & 1 & \vdots & 1 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

1c: We can continue row operations to obtain the reduced form:

$$\begin{bmatrix} 1 & -\frac{3}{2} & -\frac{1}{2} & 1 & \frac{3}{2} & \vdots & 2 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{5}{2} & \vdots & -2 \\ 0 & 0 & 0 & 0 & 1 & \vdots & 1 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \xrightarrow[\mathbf{1st} = \mathbf{1st} - \frac{3}{2} \times \mathbf{3rd}]{\mathbf{2nd} = \mathbf{2nd} - \frac{5}{2} \times \mathbf{3rd}} \begin{bmatrix} 1 & -\frac{3}{2} & -\frac{1}{2} & 1 & 0 & \vdots & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & 0 & 0 & \vdots & -\frac{9}{2} \\ 0 & 0 & 0 & 0 & 1 & \vdots & 1 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & -\frac{3}{2} & -\frac{1}{2} & 1 & 0 & \vdots & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & 0 & 0 & \vdots & -\frac{9}{2} \\ 0 & 0 & 0 & 0 & 1 & \vdots & 1 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \xrightarrow{\mathbf{1st} = \mathbf{1st} + \frac{3}{2} \times \mathbf{2nd}} \begin{bmatrix} 1 & 0 & -\frac{1}{4} & 1 & 0 & \vdots & \frac{25}{4} \\ 0 & 1 & \frac{1}{2} & 0 & 0 & \vdots & -\frac{9}{2} \\ 0 & 0 & 0 & 0 & 1 & \vdots & 1 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

1d: The rank of the matrix is equal to the number of linearly independent row. Hence, the rank of the augmented matrix is 3.

1e: The lead variables are x_1, x_2, and x_5 whereas x_3 and x_4 are free variables.

1f: The general solution is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} \frac{25}{4} \\ -\frac{9}{2} \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

2 Determinants (10 pts)

Find the determinant of the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{bmatrix}.$$

Required Knowledge: **Determinants, row/column operations.**

Solution:

Direct calculations yield:

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 3 & 8 & 15 \\ 0 & 7 & 26 & 63 \end{pmatrix} \quad \text{[row operations of type III]}$$

$$= \det \begin{pmatrix} 1 & 2 & 3 \\ 3 & 8 & 15 \\ 7 & 26 & 63 \end{pmatrix} \quad \text{[expansion w.r.t. the first row]}$$

$$= \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 6 \\ 0 & 12 & 42 \end{pmatrix} \quad \text{[row operations of type III]}$$

$$= \det \begin{pmatrix} 2 & 6 \\ 12 & 42 \end{pmatrix} = 2 \cdot 3 \cdot \det \begin{pmatrix} 1 & 3 \\ 4 & 14 \end{pmatrix} = 6 \cdot (14 - 12) = 12.$$

3 Vectorspaces (4 + 8 + 8 = 20 pts)

Let V be the vector space of functions of the form

$$ae^x + bxe^x + cx^2e^x$$

where a, b, and c are scalars. Let $L : V \to V$ be given by

$$L(f) = f + f'.$$

(a) Is L a linear transformation?

(b) Find the matrix representing L with respect to the basis $\{e^x, xe^x, x^2e^x\}$.

(c) Using the matrix representation obtained in **??**, find the solution of the differential equation:

$$f(x) + f'(x) = 2(1 + x + x^2)e^x.$$

Required Knowledge: **Vector spaces, subspaces, linear transformations, matrix representations.**

Solution:

3a: Let α be a scalar and $f \in V$. Then, we have $L(\alpha f) = (\alpha f) + (\alpha f)' = \alpha(f + f') = \alpha L(f)$. Now, let $g \in V$. Then, we have $L(f + g) = (f + g) + (f + g)' = (f + f') + (g + g') = L(f) + L(g)$. Therefore, L is a linear transformation.

3b: To find the matrix representation, we apply L to the basis vectors:

$$\begin{aligned} L(e^x) &= e^x + (e^x)' = 2e^x = 2 \cdot e^x + 0 \cdot xe^x + 0 \cdot x^2e^x \\ L(xe^x) &= xe^x + (xe^x)' = xe^x + e^x + xe^x = e^x + 2xe^x = 1 \cdot e^x + 2 \cdot xe^x + 0 \cdot x^2e^x \\ L(x^2e^x) &= x^2e^x + (x^2e^x)' = x^2e^x + 2xe^x + x^2e^x = 2xe^x + 2x^2e^x = 0 \cdot e^x + 2 \cdot xe^x + 2 \cdot x^2e^x. \end{aligned}$$

Therefore, the matrix representing L is

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

3c: The differential equation

$$f(x) + f'(x) = 2(1 + x + x^2)e^x$$

is equivalent to the linear system of equations

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

This leads to the solution $c = 1$, $b = 0$, and $a = 1$. Consequently, the solution of the differential equation is given by

$$f(x) = e^x + x^2e^x.$$

4 Least squares problem (15 pts)

Find the ellipse $a^2x^2 + b^2y^2 = 1$ that gives the best least squares approximation to the points:

x	-2	0	0	2
y	0	$-\sqrt{2}$	1	0

REQUIRED KNOWLEDGE: **Least-squares problem, normal equations.**

SOLUTION:

Let $\alpha = a^2$ and $\beta = b^2$. Then, we want to solve the least squares problem:

$$\begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 0 & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

The corresponding normal equations are given by:

$$\begin{bmatrix} 32 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}.$$

Therefore, we obtain $\alpha = \frac{1}{4}$ and $\beta = \frac{3}{5}$. Consequently, $a = \pm\frac{1}{2}$ and $b = \pm\frac{\sqrt{3}}{\sqrt{5}}$.

5 Characteristic polynomial (2 + 2 + 2 + 2 + 2 = 10 pts)

Let $A \in \mathbb{R}^{4\times 4}$ and $\det(A - \lambda I) = \lambda^4 - 5\lambda^2 + 4$.

(a) Determine the determinant of A.

(b) Is A nonsingular? Why?

(c) Determine rank(A).

(d) Find all eigenvalues of A.

(e) Is A diagonalizable?

REQUIRED KNOWLEDGE: **Characteristic polynomial, eigenvalues, determinant, diagonalizability.**

SOLUTION:

5a: $\det(A) = \det(A - 0 \cdot I) = 4$.

5b: Since $\det(A) \neq 0$, A is nonsingular.

5c: Since A is nonsingular, its columns are linearly independent and hence rank$(A) = 4$.

5d: Eigenvalues are roots of the characteristic polynomial. Note that $\lambda^4 - 5\lambda^2 + 4 = (\lambda^2 - 1)(\lambda^2 - 4)$. Therefore, eigenvalues of A are $\lambda_{1,2} = \mp 1$ and $\lambda_{3,4} = \mp 2$.

5e: Since all eigenvalues are distinct, it is diagonalizable.

6 Eigenvalues/vectors ((3 + 6 + 6) + 5 = 20 pts)

(a) Consider the matrix

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \\ 2 & -2 & -1 \end{bmatrix}.$$

(i) Show that -3 and 3 are its eigenvalues.

(ii) Find its eigenvectors corresponding to these eigenvalues.

(iii) Is it diagonalizable? If so, find a diagonalizer.

(b) By using the definition of matrix exponential, find e^A where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Required Knowledge: **Eigenvalues, eigenvectors, and diagonalization, matrix exponential.**

Solution:

6a(i): We know that λ is an eigenvalue of A if and only if $A - \lambda I$ is singular. Therefore, -3 is an eigenvalue if and only if

$$\det \begin{pmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{pmatrix} = 0.$$

Note that

$$\begin{aligned} \det \begin{pmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{pmatrix} &= \det \begin{pmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \\ 2 & -2 & 2 \end{pmatrix} && \text{[row operations]} \\ &= 0. && \text{[first two rows are identical]} \end{aligned}$$

As such, -3 is an eigenvalue.

For 3, we need to check that whether

$$\begin{bmatrix} -1 & 1 & 2 \\ 1 & -1 & -2 \\ 2 & -2 & -4 \end{bmatrix}$$

is singular. Since the third row is twice the second, we have

$$\det \begin{pmatrix} -1 & 1 & 2 \\ 1 & -1 & -2 \\ 2 & -2 & -4 \end{pmatrix} = 0$$

and hence 3 is an eigenvalue.

The matrix A has an eigenvalue λ if and only if $\det(A - \lambda I) = 0$. Note that

$$\det(A - I) = \det(\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & -1 \end{bmatrix}) = 0$$

as the first column consists entirely of zeros. Therefore, 1 is an eigenvalue of A.

6b: Note that

$$\det(A-\lambda I)=\det(\begin{bmatrix}1-\lambda & 1 & 1\\ 0 & -\lambda & 1\\ 0 & -1 & -\lambda\end{bmatrix})=(1-\lambda)\det(\begin{bmatrix}-\lambda & 1\\ -1 & -\lambda\end{bmatrix})=(1-\lambda)(\lambda^2+1).$$

This results in the eigenvalues $\lambda_1=1$, $\lambda_{2,3}=\pm i$.

6c: In order to find a diagonalizer, we need to compute an eigenvector for each eigenvalue:

(a) $\lambda_1=1$: Note that

$$0=(A-I)x_1=\begin{bmatrix}0 & 1 & 1\\ 0 & -1 & 1\\ 0 & -1 & -1\end{bmatrix}x_1$$

results in

$$x_1=\begin{bmatrix}1\\0\\0\end{bmatrix}.$$

(b) $\lambda_2=i$: Note that

$$0=(A-iI)x_2=\begin{bmatrix}1-i & 1 & 1\\ 0 & -i & 1\\ 0 & -1 & -i\end{bmatrix}x_2$$

results in

$$x_2=\begin{bmatrix}-i\\1\\i\end{bmatrix}.$$

(c) $\lambda_3=-i$: Note that

$$0=(A-iI)x_3=\begin{bmatrix}1+i & 1 & 1\\ 0 & i & 1\\ 0 & -1 & i\end{bmatrix}x_3$$

results in

$$x_3=\begin{bmatrix}i\\1\\-i\end{bmatrix}.$$

Therefore, one can take

$$T=\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}=\begin{bmatrix}1 & -i & i\\ 0 & 1 & 1\\ 0 & i & -i\end{bmatrix}.$$

To verify that T is a diagonalizer, note that

$$AT=\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 1\\ 0 & -1 & 0\end{bmatrix}\begin{bmatrix}1 & -i & i\\ 0 & 1 & 1\\ 0 & i & -i\end{bmatrix}=\begin{bmatrix}1 & 1 & 1\\ 0 & i & -i\\ 0 & -1 & -1\end{bmatrix}$$

and

$$T\begin{bmatrix}1 & 0 & 0\\ 0 & i & 0\\ 0 & 0 & -i\end{bmatrix}=\begin{bmatrix}1 & -i & i\\ 0 & 1 & 1\\ 0 & i & -i\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & i & 0\\ 0 & 0 & -i\end{bmatrix}=\begin{bmatrix}1 & 1 & 1\\ 0 & i & -i\\ 0 & -1 & -1\end{bmatrix}.$$
